電磁気学 B 演習　第 5 回解答

1.　$\mathbf{B}=\nabla\times\mathbf{A}$ に $\mathbf{A}$ を代入して計算する。

(1) $\mathbf{B}=\nabla\times\frac{1}{c}(\mathbf{c}\times\mathbf{r})=\frac{1}{c}\nabla\times(\mathbf{c}\times\mathbf{r})=\frac{1}{c}\{\mathbf{c}(\nabla\cdot\mathbf{r})-(\mathbf{c}\cdot\nabla)\mathbf{r}\}$

ここで

$$\nabla\cdot\mathbf{r}=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z=3$$

$$\therefore \mathbf{c}(\nabla\cdot\mathbf{r})=3\mathbf{c}$$

$$\mathbf{c}\cdot\nabla=(c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k})\cdot\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)=c_x\frac{\partial}{\partial x}+c_y\frac{\partial}{\partial y}+c_z\frac{\partial}{\partial z}$$

$$\therefore (\mathbf{c}\cdot\nabla)\mathbf{r}=\left(c_x\frac{\partial}{\partial x}+c_y\frac{\partial}{\partial y}+c_z\frac{\partial}{\partial z}\right)\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}=\mathbf{c}$$

$$\therefore \mathbf{B}=\frac{1}{c}\{\mathbf{c}(\nabla\cdot\mathbf{r})-(\mathbf{c}\cdot\nabla)\mathbf{r}\}=\frac{1}{c}\{3\mathbf{c}-\mathbf{c}\}=\frac{2\mathbf{c}}{c}$$

(2) $\mathbf{B}=\nabla\times c\mathbf{r}=c\nabla\times\mathbf{r}=c\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=0$

(3) $\mathbf{B}=\nabla\times\frac{\mathbf{c}\times\mathbf{r}}{r^3}=\mathbf{c}\left(\nabla\cdot\frac{\mathbf{r}}{r^3}\right)-(\mathbf{c}\cdot\nabla)\frac{\mathbf{r}}{r^3}$

ここで

$$\begin{aligned}\nabla\cdot\frac{\mathbf{r}}{r^3}&=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3}\right)\\&=\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\\&=\frac{r^2-3x^2}{r^5}+\frac{r^2-3y^2}{r^5}+\frac{r^2-3z^2}{r^5}\\&=\frac{3r^2-3(x^2+y^2+z^2)}{r^5}=0\end{aligned}$$

$$\left(\because \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^{\frac{6}{2}}}=\frac{r^3-3x^2r}{r^6}=\frac{r^2-3x^2}{r^5}\right)$$

$$(\mathbf{c}\cdot\nabla)\frac{\mathbf{r}}{r^3}=\frac{1}{r^3}(\mathbf{c}\cdot\nabla)\mathbf{r}+\mathbf{r}\left(\mathbf{c}\cdot\nabla\frac{1}{r^3}\right)=\frac{\mathbf{c}}{r^3}-\frac{3(\mathbf{c}\cdot\mathbf{r})}{r^5}\mathbf{r}$$

$$\left(\begin{array}{l} \because \dfrac{\partial}{\partial x}\left(\dfrac{1}{r^3}\right) = \dfrac{-\dfrac{3}{2}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\cdot 2x}{\left(x^2+y^2+z^2\right)^{\frac{6}{2}}} = -\dfrac{3x}{r^5} \\ \nabla\dfrac{1}{r^3} = -\dfrac{3}{r^5}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right) = -\dfrac{3}{r^5}\mathbf{r} \end{array}\right)$$

以上より

$$\mathbf{B} = -\frac{\mathbf{c}}{r^3} + \frac{3(\mathbf{c}\cdot\mathbf{r})}{r^5}\mathbf{r}$$

(4) $\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left(-a\mathbf{k}\right) = -\dfrac{\partial}{\partial y}a\mathbf{i} + \dfrac{\partial}{\partial x}a\mathbf{j}$ 、 $a = \dfrac{\mu_0 I}{2\pi}\ln\left(\dfrac{r}{l}\right)$ とおく

ここで

$$\frac{\partial}{\partial x}a = \frac{\partial}{\partial x}\left\{\frac{\mu_0 I}{2\pi}\ln\left(\frac{r}{l}\right)\right\} = \frac{\mu_0 I}{2\pi}\cdot\frac{x}{r^2}$$

よって

$$\mathbf{B} = -\frac{\partial}{\partial y}a\mathbf{i} + \frac{\partial}{\partial x}a\mathbf{j} = -\frac{\mu_0 I}{2\pi}\cdot\frac{y}{r^2}\mathbf{i} + \frac{\mu_0 I}{2\pi}\cdot\frac{x}{r^2}\mathbf{j} = -\frac{\mu_0 I}{2\pi r^2}\left(y\mathbf{i}-x\mathbf{j}\right)$$

2. 準備中

3. (1) 表面電流密度 $\mathbf{j} = \sigma\mathbf{v} = \sigma v\mathbf{z}$

(2) ベクトルポテンシャルは

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}}{r}ds = \frac{\mu_0}{4\pi}\sigma v\mathbf{z}\int\frac{ds}{r}$$

ベルトが静止している場合の静電ポテンシャルは

$$\Phi(\mathbf{r}) = \frac{\sigma}{4\pi\varepsilon_0}\int\frac{ds}{r}$$

これより

$$\mathbf{A}(\mathbf{r}) = \mu_0\varepsilon_0 v\mathbf{z}\Phi(\mathbf{r})$$

(3) 磁場 $\mathbf{B}$ と電場 $\mathbf{E}$ はそれぞれ

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla\Phi$$

であるから、したがって

$$\mathbf{B}(\mathbf{r}) = \mu_0\varepsilon_0\nabla\times(\mathbf{v}\Phi) = \mu_0\varepsilon_0\mathbf{v}\times(-\nabla\Phi) = \mu_0\varepsilon_0\mathbf{v}\times\mathbf{E}(\mathbf{r})$$

4. 任意の点 P をとおり導線に垂直な平面が両線と交わる点を結ぶ線を x 軸、その中点 O を原点とし、導線と平行に z 軸、それらに垂直に y 軸をとる。導線の線要素 ds は xy 平面

方向の成分を持たないから $A_x = A_y = 0$

$$A_z = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{z^2+r_1^2}}-\frac{1}{\sqrt{z^2+r_2^2}}\right)dz$$

$$= \frac{\mu_0 I}{2\pi}[\log(z+\sqrt{z^2+r_1^2})-\log(z+\sqrt{z^2+r_2^2})]_0^{\infty}$$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{\log(z+\sqrt{z^2+r_1^2})}{\log(z+\sqrt{z^2+r_2^2})}\right]_0^{\infty}$$

$$= -\frac{\mu_0 I}{2\pi}\log\frac{r_1}{r_2}$$

$$r_1^2 = (x+d)^2 + y^2, r_2^2 = (x-d)^2 + y^2$$

であるから

$$B_x = (\nabla\times\mathbf{A})_x = \frac{\partial A_z}{\partial y}$$

$$= \frac{\mu_0 I}{2\pi}\left\{\frac{y}{(x-d)^2+y^2}+\frac{y}{(x+d)^2+y^2}\right\}$$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{y}{r_2^2}-\frac{y}{r_1^2}\right) = \frac{\mu_0 I}{2\pi}\left(-\frac{\sin\theta_1}{r_1}+\frac{\sin\theta_2}{r_2}\right)$$

$$B_y = (\nabla\times\mathbf{A})_y = -\frac{\partial A_z}{\partial y}$$

$$= \frac{\mu_0 I}{2\pi}\left\{\frac{(x+d)}{(x+d)^2+y^2}-\frac{(x-d)}{(x-d)^2+y^2}\right\}$$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{x+d}{r_1^2}-\frac{x-d}{r_2^2}\right) = \frac{\mu_0 I}{2\pi}\left(\frac{\cos\theta_1}{r_1}-\frac{\cos\theta_2}{r_2}\right)$$

$$B_z = (\nabla\times\mathbf{A})_z = 0$$

これより、求める磁場 B は

$$B = \sqrt{(B_x^2+B_y^2)} = \frac{\mu_0 I}{2\pi}\sqrt{\left(-\frac{\sin\theta_1}{r_1}+\frac{\sin\theta_2}{r_2}\right)^2+\left(\frac{\cos\theta_1}{r_1}-\frac{\cos\theta_2}{r_2}\right)^2}$$

$$= \frac{\mu_0 I}{2\pi}\sqrt{\frac{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)}{r_1^2r_2^2}} = \frac{\mu_0 I}{\pi r_1 r_2}$$