電磁気学要論　演習
第4回　解答

1. (1)　x 成分についてのみ考える。

$$
\begin{aligned}
(\text{左辺})_x &= \frac{\partial}{\partial y}(\nabla \times \mathbf{A})_z - \frac{\partial}{\partial z}(\nabla \times \mathbf{A})_y \\
&= \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\
&= \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_y}{\partial y}\right) + \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \\
&= \{\nabla(\nabla \cdot \mathbf{A})\}_x - \left\{\nabla^2 \mathbf{A}\right\}_x = (\text{右辺})_x
\end{aligned}
$$

y, z 成分についても同様で、左辺=右辺

(2)　x 成分についてのみ考える。

$$
\begin{aligned}
(\text{左辺})_x &= A_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial A_x}{\partial z} \\
(\text{右辺})_x &= \frac{1}{2}\frac{\partial}{\partial x}(A_x^2 + A_y^2 + A_z^2) - \{A_y(\nabla \times \mathbf{A})_z - A_z(\nabla \times \mathbf{A})_y\} \\
&= A_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial A_y}{\partial x} + A_z \frac{\partial A_z}{\partial x} - \left\{A_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - A_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\} \\
&= A_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial A_x}{\partial z} = (\text{左辺})_x
\end{aligned}
$$

y, z 成分についても同様で、左辺=右辺

2. (1)　線分 OA の方程式は媒介変数 u を用いて

$$x = u, y = 2u, z = 3u, (0 \leq u \leq 1)$$

とおける。ここで、

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{14}du$$

よって求める積分は

$$\int_{\mathrm{OA}} f(x,y,z)ds = \int_0^1 (2u\cdot 3u + 3u\cdot u + u\cdot 2u)\sqrt{12}du = \frac{11\sqrt{14}}{3}$$

(2) OB 上では $0 \leq x \leq 1, y = z = 0$、BC 上では $x = 1, 0 \leq y \leq 2, z = 0$、CA 上では $x = 1, y = 2, 0 \leq z \leq 3$ であるから、

$$\begin{aligned}\int_{\mathrm{OA}} f(x,y,z)ds &= \int_0^1 f(x,0,0)dx + \int_0^2 f(1,y,0)dy + \int_0^3 f(1,2,z)dz \\ &= \frac{43}{2}\end{aligned}$$

3. 曲線 $y = x^3$ に沿って接線線積分を求める。

$$\int \mathbf{A}\cdot d\mathbf{r} = \int (xydx - xdy) = \int (x\cdot x^3dx - x\cdot 3x^2dx) = -\frac{11}{20}$$

つぎに、OB 上では $\mathbf{A} = -x\mathbf{j}, d\mathbf{r} = \mathbf{i}dx$、BA 上では $\mathbf{A} = y\mathbf{i} - \mathbf{j}, d\mathbf{r} = \mathbf{j}dy$ であるから、

$$\int_{\mathrm{OB}} \cdots + \int_{\mathrm{BA}} \cdots = \int_0^1 (-x\mathbf{j})\cdot\mathbf{i}dx + \int_0^1 (y\mathbf{i} - \mathbf{j})\cdot\mathbf{j}dy = -1$$

4. (1) 平面の方程式は $z = 2 - 2x - 2y$。

$$\frac{\partial z}{\partial x} = -2, \frac{\partial z}{\partial y} = -2$$

であるから、

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = 3$$

よって $dS = 3dxdy$。

$$f(x,y,z) = x^2 + 3y + (2 - 2x - 2y) - 1 = (x-1)^2 + y$$

として、

$$\int_S f(x,y,z)dS = \int_{S'} (x-1)^2 + y \cdot 3dxdy$$
$$= 3\int_0^1 \left\{ \int_0^{1-x} (x-1)^2 + y \cdot dy \right\} dx$$
$$= 3\int_0^1 -(x-1)^3 + \frac{1}{2}(x-1)^2 \cdot dx = \frac{5}{4}$$

(2)　$\mathbf{n}$ の成分は

$$\frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\left(-\frac{\partial z}{\partial x}\right) = \frac{2}{3}$$

$$\frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\left(-\frac{\partial z}{\partial y}\right) = \frac{2}{3}$$

$$\frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} = \frac{1}{3}$$

これから

$$\mathbf{A}\cdot\mathbf{n} = \frac{1}{3}(2x^2+z) = \frac{2}{3}(x^2-x-y+1)$$

として、

$$\int_S \mathbf{A}\cdot\mathbf{n}dS = \int_{S'} \frac{2}{3}(x^2-x-y+1)\cdot 3dxdy = 2\int_0^1 \left\{ \int_0^{1-y} (x^2-x-y+1)dx \right\} dy$$
$$= 2\int_0^1 \left\{ \frac{1}{3}(1-x)^3 + \frac{1}{2}(1-y)^2 \right\} dy = \frac{1}{2}$$