電磁気学要論演習第三回解答

1.　(a) $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

よって P(-1,1,1) における勾配は $\mathbf{i} - \mathbf{j} - \mathbf{k}$ となる。

(c) $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} = yz\cos(xyz)\mathbf{i} + xz\cos(xyz)\mathbf{j} + xy\cos(xyz)\mathbf{k}$

よって P(-1,1,1) における勾配は $\dfrac{\pi}{6}\mathbf{i} + \dfrac{1}{6}\mathbf{j} + \dfrac{\pi}{2}\mathbf{k}$ となる。

(d) $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k} = -2e^{-2x}\cos(yz)\mathbf{i} - ze^{-2x}\sin(yz)\mathbf{j} - ye^{-2x}\sin(yz)\mathbf{k}$

よって P$(0,\pi,1/4)$ における勾配は $-\sqrt{2}\mathbf{i} - \dfrac{\sqrt{2}}{8}\mathbf{j} - \dfrac{\sqrt{2}\pi}{2}\mathbf{k}$ となる。

2.　(a) $\boldsymbol{A} = (a_1, a_2, a_3)$、$\boldsymbol{B} = (b_1, b_2, b_3)$ とする。

$$\begin{aligned}
\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \nabla\cdot\{(a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}\} \\
&= \frac{\partial}{\partial x}(a_2b_3 - a_3b_2) + \frac{\partial}{\partial y}(a_3b_1 - a_1b_3) + \frac{\partial}{\partial z}(a_1b_2 - a_2b_1) \\
&= a_2\frac{\partial b_3}{\partial x} + b_3\frac{\partial a_2}{\partial x} - a_3\frac{\partial b_2}{\partial x} - b_2\frac{\partial a_3}{\partial x} + a_3\frac{\partial b_1}{\partial y} + b_1\frac{\partial a_3}{\partial y} \\
&\quad - a_1\frac{\partial b_3}{\partial y} - b_3\frac{\partial a_1}{\partial y} + a_1\frac{\partial b_2}{\partial z} + b_2\frac{\partial a_1}{\partial z} - a_2\frac{\partial b_1}{\partial z} - b_1\frac{\partial a_2}{\partial z} \qquad (2.1)
\end{aligned}$$

一方、

$$\begin{aligned}
\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) &= \boldsymbol{B}\cdot\left\{\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\boldsymbol{k}\right\} \\
&\quad - \boldsymbol{A}\cdot\left\{\left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial b_1}{\partial z} - \frac{\partial b_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y}\right)\boldsymbol{k}\right\} \\
&= b_1\frac{\partial a_3}{\partial y} - b_1\frac{\partial a_2}{\partial z} + b_2\frac{\partial a_1}{\partial z} - b_2\frac{\partial a_3}{\partial x} + b_3\frac{\partial a_2}{\partial x} - b_3\frac{\partial a_1}{\partial y} \\
&\quad - a_1\frac{\partial b_3}{\partial y} + a_1\frac{\partial b_2}{\partial z} - a_2\frac{\partial b_1}{\partial z} + a_2\frac{\partial b_3}{\partial x} - a_3\frac{\partial b_2}{\partial x} + a_3\frac{\partial b_1}{\partial y} \qquad (2.2)
\end{aligned}$$

式(2.1)と式(2.2)より、

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$$

が示された。

(b) $\nabla \times (\nabla \varphi) = \nabla \times \left(\frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j} + \frac{\partial \varphi}{\partial z}\boldsymbol{k} \right)$

$$= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right)\boldsymbol{i} + \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right)\boldsymbol{j} + \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right)\boldsymbol{k}$$

$$= 0$$

(c) $\nabla \cdot (\nabla \times \boldsymbol{A}) = \nabla \cdot \left\{ \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)\boldsymbol{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right)\boldsymbol{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)\boldsymbol{k} \right\}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

$$= \frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z} + \frac{\partial^2 a_1}{\partial y \partial z} - \frac{\partial^2 a_3}{\partial y \partial x} + \frac{\partial^2 a_2}{\partial z \partial x} - \frac{\partial^2 a_1}{\partial z \partial y}$$

$$= 0$$

3. (a) $\nabla \varphi \cdot \mathbf{A} = \left\{ \frac{\partial}{\partial x}(2x^2yz^3)\mathbf{i} + \frac{\partial}{\partial y}(2x^2yz^3)\mathbf{j} + \frac{\partial}{\partial z}(2x^2yz^3)\mathbf{k} \right\} \cdot (2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k})$

$$= (4xyz^3\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}) \cdot (2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k})$$

$$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$$

(b) $\varphi \nabla \cdot \mathbf{A} = 2x^2yz^3 \left\{ \frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(xz^2) \right\}$

$$= 2x^2yz^3 \cdot (-x^2 + 2xz)$$

$$= -2x^4yz^3 + 4x^3yz^4$$

(c) $\nabla \cdot (\varphi \mathbf{A}) = \nabla \cdot (4x^2y^2z^4\mathbf{i} - 2x^4y^2z^3\mathbf{j} + 2x^3yz^5\mathbf{k})$

$$= \frac{\partial}{\partial x}(4x^2y^2z^4) - \frac{\partial}{\partial y}(2x^4y^2z^3) + \frac{\partial}{\partial z}(2x^3yz^5)$$

$$= 8xy^2z^4 - 4x^4yz^3 + 10x^3yz^4$$

(a)〜(c)の結果より、$\nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A} = \nabla \cdot (\varphi \mathbf{A})$であることが確認できる。

(d) $(\mathbf{B} \cdot \nabla)\mathbf{A} = \left(x^2 \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} - xy \frac{\partial}{\partial z} \right)(2yz\mathbf{i} - x^2y\mathbf{j} + xz^2\mathbf{k})$

$$= (2yz^2 - 2xy^2)\mathbf{i} - (2x^3y + x^2yz)\mathbf{j} + (x^2z^2 - 2x^2yz)\mathbf{k}$$

(e)

$$(\mathbf{A}\times\nabla)\varphi=\left\{\left(-x^2y\frac{\partial}{\partial z}-xz^2\frac{\partial}{\partial y}\right)\mathbf{i}+\left(xz^2\frac{\partial}{\partial x}-2yz\frac{\partial}{\partial z}\right)\mathbf{j}+\left(2yz\frac{\partial}{\partial y}+x^2y\frac{\partial}{\partial x}\right)\mathbf{k}\right\}2x^2yz^3$$

$$=-\left(6x^4y^2z^2+2x^3z^5\right)\mathbf{i}+\left(4x^2yz^5-12x^2y^2z^3\right)\mathbf{j}+\left(4x^2yz^4+4x^3y^2z^3\right)\mathbf{k}$$

(f) $\mathbf{A}\times\nabla\varphi=\left(2yzi-x^2yj+xz^2k\right)\times\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)2x^2yz^3$

$$=\left(2yz\mathbf{i}-x^2y\mathbf{j}+xz^2\mathbf{k}\right)\times\left(4xyz^3\mathbf{i}+2x^2z^3\mathbf{j}+6x^2yz^2\mathbf{k}\right)$$
$$=-\left(6x^4y^2z^2+2x^3z^5\right)\mathbf{i}+\left(4x^2yz^5-12x^2y^2z^3\right)\mathbf{j}+\left(4x^2yz^4+4x^3y^2z^3\right)\mathbf{k}$$

(e)、(f)の結果より $(\mathbf{A}\times\nabla)\varphi=\mathbf{A}\times\nabla\varphi$ であることが確認できる。

4. $\nabla\cdot\boldsymbol{A}=z^3-2x^2z+8yz^3$

$\nabla\times\boldsymbol{A}=\left(2z^4+2x^2y\right)\boldsymbol{i}+3xz^2\boldsymbol{j}-4xyz\boldsymbol{k}$

よって、点 P(1,-1,1) においては $\nabla\cdot\boldsymbol{A}=-9$、$\nabla\times\boldsymbol{A}=3\boldsymbol{j}+4\boldsymbol{k}$ となる。